DISCRIMINANTS AND JACOBIANS OF VIRTUAL REFLECTION GROUPS

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Abstract. Let A be a polynomial algebra with complex coefficients. Let B be a finite extension ring of A which is also a polynomial algebra. We describe the factorisation of the Jacobian J of the extension into irreducibles. We also introduce the notion of a well-ramified extension and define its discriminant polynomial D. In the particular case where A is the ring of invariants of B under the action of a group (*i.e.*, a Galois extension), this framework corresponds to the classical invariant theory of complex reflection groups. In the more general case of a well-ramified extension, we explain how the pair (D, J) behaves similarly to a Galois extension. This work can be viewed as the first step towards a possible invariant theory of "virtual reflection groups".

Introduction

This note describes a non-Galois version of the first few steps of the classical invariant theory of reflection groups. We will deal with quite basic questions of commutative algebra, that were at first motivated by empirical observations on the extensions defined by Lyashko-Looijenga morphisms.

We consider a finite ring extension $A \subseteq B$, where A and B are both polynomial algebras in n variables. In the case when A is the ring of invariants of B under the action of a group G, this implies that G is generated by reflections (Chevalley-Shephard-Todd's theorem) and many properties are known in this setting. Here we do not suppose that A has the form B^G. Thus, we cannot simply imitate the classical proofs of invariant theory, as they really make use of the group action. However, in our setting, many properties seem to work the same way as for Galois extensions, particularly for Jacobian and "discriminant" of the extension.

Note that we use only elementary commutative algebra, and that the properties derived here are presumably folklore. The situation that we describe is in fact surprisingly basic and universal, yet apparently written nowhere from this perspective. The extensions usually studied in the litterature are either much more general, or of the form $A = B^G \subseteq B$, where B is a polynomial algebra; here we are rather interested in extensions $A \subseteq B$ where B *and* A are polynomial algebras, but where we *do not require* A to be the ring of invariants of B under a group action.

The key ingredients to describe and understand the situation are:

- a notion of *"well-ramified"* polynomial extensions;
- properties of the *different* of an extension, that enable to apprehend the Jacobian of the extension.

Motivations. Let V be an n-dimensional complex vector space, and $W \subseteq \mathrm{GL}(V)$ a finite complex reflection group, with fundamental system of invariants $f_1, \ldots, f_n$ of degrees $d_1 \leq \cdots \leq d_n$. From Chevalley-Shephard-Todd's theorem, we have the equality $\mathbb{C}[V]^W = \mathbb{C}[f_1, \ldots, f_n]$, and the isomorphism

$$\begin{array}{lcl} W\backslash V & \to & \mathbb{C}^n \\ \bar{v} & \mapsto & (f_1(v), \ldots, f_n(v)). \end{array}$$

Let us denote by $\mathcal{A}$ the set of all reflecting hyperplanes, and consider the discriminant of W defined by

$$\Delta_W := \prod_{H \in \mathcal{A}} \alpha_H^{e_H} \ ,$$

where α_H is an equation of H and e_H is the order of the parabolic subgroup W_H. The discriminant is the equation of the hypersurface $\mathcal{H} := W\backslash \bigcup_{H \in \mathcal{A}} H$ in $\mathbb{C}^n = \operatorname{Spec} \mathbb{C}[f_1, \ldots, f_n]$.

Let us also consider the Jacobian J_W of the morphism $(v_1, \ldots, v_n) \mapsto (f_1(v), \ldots, f_n(v))$:

$$J_W := \det \left(\frac{\partial f_i}{\partial v_j} \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} .$$

It is well known (see for example [Kan01, Sect. 21]) that the Jacobian satisfies the following factorisation:

$$J_W \doteq \prod_{H \in \mathcal{A}} \alpha_H^{e_H - 1},$$

where $\doteq$ denotes equality up to a nonzero scalar. Thus, we get

$$\Delta_W / J_W = \prod_{H \in \mathcal{A}} \alpha_H \ ,$$

i.e. this quotient is the product of the ramified polynomial of the extension $\mathbb{C}[f_1, \ldots, f_n] \subseteq \mathbb{C}[V]$.

A stunningly similar situation has came up recently, in the geometric study of the Lyashko-Looijenga morphisms (LL) associated to complex reflection groups. These morphisms have been introduced by Bessis in his work about the $K(\pi, 1)$ property for finite complex reflection arrangements [Bes07]; see also [Rip10c] about the relation with the combinatorics of factorisations of Coxeter elements. The Lyashko-Looijenga morphism associated to a rank n reflection group has the following form:

$$\begin{array}{cccc} \mathrm{LL} & \mathbb{C}^{n-1} & \to & \mathbb{C}^{n-1} \\ & X = (x_1, \ldots, x_n) & \mapsto & (a_2(X), \ldots, a_n(X)) \ , \end{array}$$

where $a_2, \ldots, a_n$ are polynomials in $x_1, \ldots, x_{n-1}$, constructed from the geometry of the reflection group. We refer to [Bes07, Sect. 5] or [Rip10c, Sect. 3] for the precise definition (which is not necessary in this note).

Associated to the morphism LL, we can naturally define the LL-discriminant D_{LL} (it is the equation of the bifurcation locus LL), and the LL-Jacobian J_{LL} (the Jacobian determinant of the morphism). It turns out that the pair of polynomials $(J_{\mathrm{LL}}, D_{\mathrm{LL}})$ behaves similarly to the pair (J_W, Δ_W) defined earlier: the quotient $D_{\mathrm{LL}}/J_{\mathrm{LL}}$ is the product of the ramified polynomials of the extension associated to LL, and moreover their valuations in D_{LL} correspond to their ramification indices.

General setting and main result. Although we first had in mind applications to the Lyashko-Looijenga context, it seems that a more general setting has its interest itself. Thus, this note is devoted to the following general setup. Let us consider a *finite graded polynomial extension* $A \subseteq B$ (see Def. 1.1): we have a graded polynomial algebra B in n indeterminates over $\mathbb{C}$, and a polynomial subalgebra A generated by n weighted homogeneous elements of B, such that the extension is finite. The two key examples are the ones already mentioned:

- the Galois extensions $\mathbb{C}[f_1, \dots, f_n] \subseteq \mathbb{C}[v_1, \dots, v_n]$, defined by a morphism $V \to W \backslash V$, where w is a reflection group and $\mathbb{C}[f_1, \dots, f_n] = \mathbb{C}[V]^W$;
- the Lyashko-Looijenga extensions $\mathbb{C}[a_2, \dots, a_n] \subseteq \mathbb{C}[x_1, \dots, x_{n-1}]$, given by a morphism LL; these extensions are indeed finite according to [Bes07, Thm. 5.3].

In the first section we give the precise definitions, and use the notion of *different ideal* of an extension to describe a factorisation of the Jacobian. In section 2, about the geometry of such extensions, we recall the relations between the ramification locus and the branch locus of a branched covering. In section 3 we define the *well-ramified property* for a finite graded polynomial extension (Def. 3.1), and we give several characterisations of this property (Prop. 3.2): this is a slightly weaker property than the normality of the extension, and is also equivalent to the equality between the preimage of the branch locus and the ramification locus.

The main result of this chapter is:

Theorem 0.1. *Let $\underline{W} = (A \subseteq B)$ be a finite graded polynomial extension. Then the Jacobian J of the extension verifies:*

$$J \doteq \prod_{Q \in \operatorname{Spec}_1^{\mathrm{ram}}(B)} Q^{e_Q - 1}$$

where $\operatorname{Spec}_1^{\mathrm{ram}}(B)$ is the set of ramified polynomials in B (up to association), and the e_Q are the associated ramification indices.

Moreover, if the extension $\underline{W}$ is well-ramified *(according to Def. 3.1), then:*

$$(J) \cap A = \left(\prod_{Q \in \operatorname{Spec}_1^{\mathrm{ram}}(B)} Q^{e_Q} \right) \qquad \textit{(as an ideal of } A\textit{)}.$$

Remark 0.2. The first part of Thm. 0.1 is a rather easy consequence of commutative algebra properties. It is probably folklore, but I could not find the formula stated anywhere.

Remark 0.3. The notation $\underline{W}$ for the extension is intentionally chosen to emphasize the analogy with the case when the extension is Galois, *i.e.* when it is given by the action of a reflection group W on the polynomial algebra B. Here there is not necessarily a group acting, but some features of the Galois case remain. That is why David Bessis proposed to call the extension $\underline{W}$ a *virtual reflection group*[1]. The polynomial $\prod Q^{e_Q}$ in the theorem above could then be called the *discriminant*of the virtual reflection group W. One can wonder whether the analogies can go further, and to what extent it is possible to construct an "invariant theory" for virtual reflection groups.

[1] David Bessis, personal communication.

1. Jacobian and different of a finite graded polynomial extension

1.1. General setting and notations.
Let n be a positive integer, and denote by B the graded polynomial algebra $\mathbb{C}[X_1, \ldots, X_n]$, where $X_1, \ldots, X_n$ are indeterminates of respective weights $b_1, \ldots, b_n$.

Let us consider n weighted homogeneous polynomials $f_1, \ldots, f_n$ in B (of respective weights $a_1, \ldots, a_n$), and the resulting (quasi-homogenenous) mapping

$$\begin{array}{rccc} f: & \mathbb{C}^n & \to & \mathbb{C}^n \\ & (x_1, \ldots, x_n) & \mapsto & (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)). \end{array}$$

We denote by A the algebra $\mathbb{C}[f_1, \ldots, f_n]$, so that we have a ring extension $A \subseteq B$.

Definition 1.1. In the above situation, if B is an A-module of finite type, we will call $A \subseteq B$ a *finite graded polynomial extension*.

Remark 1.2. In this setting, the extension is finite if and only if $f^{-1}(\{0\}) = \{0\}$, because f is a quasi-homogeneous morphism (see for example [LZ04, Thm.5.1.5]). Moreover, the rank of B over A (or the *degree* of f) is then equal to $r := \prod a_i / \prod b_i$.

The algebra B is Cohen-Macaulay, and is finite as an A-module, so B is also a free A-module of finite type. Thus $A \subseteq B$ is a finite free extension of UFDs. We denote by K and L the fields of fractions of A and B. Let us recall some notations and properties about the ramification in this context (for example see [Ben93, Chap. 3]).

If $\mathfrak{q}$ is a prime ideal of B, then $\mathfrak{p} = \mathfrak{q} \cap A$ is a prime ideal of A. In this situation we say that $\mathfrak{q}$ lies *over* $\mathfrak{p}$. By the Cohen-Macaulay theorem [Ben93, Thm.1.4.4], $\mathfrak{q}$ has height one if and only if $\mathfrak{p}$ has height one. In this case, we can write $\mathfrak{q} = (Q)$ and $\mathfrak{p} = (P)$, where $P \in A$ and $Q \in B$ are irreducible, and $(Q) \cap A = (P)$.

The ramification index of $\mathfrak{q}$ over $\mathfrak{p}$ is:

$$e(\mathfrak{q}, \mathfrak{p}) := v_Q(P)$$

(we will rather write simply $e_{\mathfrak{q}}$ or e_Q).

We denote by $\mathrm{Spec}_1(B)$ (resp. $\mathrm{Spec}_1(A)$) the set of prime ideals of B (resp. A) of height one. It is also the set of irreducible polynomials in B, up to association. By abusing the notation, in indices of products, we will write "$Q \in \mathrm{Spec}_1(B)$" instead of "$\mathfrak{q} \in \mathrm{Spec}_1(B)$ and Q is one polynomial representing $\mathfrak{q}$", and "Q over P" instead of "(Q) over (P)".

We recall the following elementary property:

Lemma 1.3. *Let $(P) \in \mathrm{Spec}_1(A)$ and $(Q) \in \mathrm{Spec}_1(B)$. Then, (Q) lies over (P) if and only if Q divides P in B. Thus, for $(P) \in \mathrm{Spec}_1(A)$, we have:*

$$P \doteq \prod_{Q \text{ over } P} Q^{e_Q} \ .$$

1.2. Different ideal and Jacobian. As our setup is compatible with that of [Ben93, Part. 3.10], we can construct the *different* $\mathfrak{D}_{B/A}$ of the extension. Let us recall that it is defined from the inverse different :

$$\mathfrak{D}_{B/A}^{-1} := \{x \in L \mid \forall y \in B, \mathrm{Tr}_{L/K}(xy) \in A\} \ ,$$

where K and L are the fields of fractions of A and B, L is regarded as a finite vector space over K, and $\mathrm{Tr}_{L/K}(u)$ denotes the trace of the endomorphism $(x \mapsto ux)$.

The different $\mathfrak{D}_{B/A}$ is then by definition the inverse fractional ideal to $\mathfrak{D}_{B/A}^{-1}$. It is a (homogeneous) divisorial ideal; in our setting, as B is a UFD, $\mathfrak{D}_{B/A}$ is thus a principal ideal. We will see below that the different is simply generated by the Jacobian $J_{B/A}$ of the extension. For now, let us denote by $\theta_{B/A}$ a homogeneous generator of $\mathfrak{D}_{B/A}$.

The different satisfies the following:

Proposition 1.4 ([Ben93, Thm.3.10.2]). *If $\mathfrak{q}$ and $\mathfrak{p} = A \cap \mathfrak{q}$ are prime ideals of height one in B and A, then $e(\mathfrak{q}, \mathfrak{p}) > 1$ if and only if $\mathfrak{D}_{B/A} \subseteq \mathfrak{q}$.*

In other words: if Q is an irreducible polynomial in B, then $e_Q > 1$ if and only if Q divides $\theta_{B/A}$.

We define the set of ramified ideals:

$$\mathrm{Spec}_1^{\mathrm{ram}}(B) := \{\mathfrak{q} \in \mathrm{Spec}_1(B) \mid e_{\mathfrak{q}} > 1\},$$

which can also be seen as a system of representatives of the irreducible polynomials Q in B which are ramified over A. By the above theorem, we have:

$$\theta_{B/A} \doteq \prod_{Q \in \mathrm{Spec}_1^{\mathrm{ram}}(B)} Q^{v_Q(\theta_{B/A})} .$$

This can be refined as:

Proposition 1.5. *For all irreducible Q in B, we have: $v_Q(\theta_{B/A}) = e_Q - 1$. Thus:*

$$\theta_{B/A} \doteq \prod_{Q \in \mathrm{Spec}_1^{\mathrm{ram}}(B)} Q^{e_Q - 1}.$$

Proof. We localize at (Q) in order to obtain local Dedekind domains. Then we can use directly Prop. 13 in [Ser68, Ch. III]. $\square$

Let us show that the different is actually generated by the Jacobian determinant. For this, we need to introduce the *Kähler different* of the extension $A \subseteq B$. According to Broer [Bro06] (end of first section), when B is a polynomial algebra, the Kähler different can be defined as the ideal generated by the Jacobians of all n-tuples of elements of A, with respect to $X_1, \ldots, X_n$. But here we are in an even more specific situation, where A is also a polynomial ring. Thus, whenever we take $g_1, \ldots, g_n \in A = \mathbb{C}[f_1, \ldots, f_n]$, we have

$$\det \left(\frac{\partial g_i}{\partial X_k} \right)_{\substack{1 \le i \le n \\ 1 \le k \le n}} = \det \left(\frac{\partial g_i}{\partial f_j} \right)_{\substack{1 \le i \le n \\ 1 \le j \le n}} \det \left(\frac{\partial f_j}{\partial X_k} \right)_{\substack{1 \le j \le n \\ 1 \le k \le n}} ,$$

so the Kähler different is simply the principal ideal of B generated by the polynomial

$$J_{B/A} := \mathrm{Jac}((f_1, \ldots, f_n)/(X_1, \ldots, X_n)) = \det \left(\frac{\partial f_i}{\partial X_j} \right)_{\substack{1 \le i \le n \\ 1 \le j \le n}} .$$

Proposition 1.6. *With the hypothesis above, we have:*

$$\theta_{B/A} \doteq J_{B/A}.$$

Proof. In [Bro06], Broer studies several notions of different ideals, and proves that under certain hypothesis they are equal. We are here in the hypothesis of his Corollary 1, which states in particular that the Kähler different is equal to the different $\mathfrak{D}_{B/A}$. $\square$

Note that we use here a strong result (which applies in much more generality than what we need). It should be possible to give a simpler proof. Let us simply add here a more explicit proof of the fact that the polynomials $J_{B/A}$ and $\theta_{B/A}$ have the same degree (using a ramification formula of Benson).

Lemma 1.7. *With the hypothesis and notations above, we have:*

$$\deg(\theta_{B/A}) = \deg(J_{B/A}).$$

Proof. First we need to recall some notations for graded algebras, see [Ben93, Ch.2.4]. If A is a graded algebra, and M a graded A-module, we denote the usual Hilbert-Poincaré series of M by $\operatorname{grdim} M$ (for "graded dimension"): $\operatorname{grdim} M := \sum_k \dim M_k t^k$. If n is the Krull dimension of A, we define rational numbers $\deg(M)$ and $\psi(M)$ by the Laurent expansion about $t = 1$:

$$\operatorname{grdim}(M) = \frac{\deg(M)}{(1-t)^n} + \frac{\psi(M)}{(1-t)^{n-1}} + o\left(\frac{1}{(1-t)^{n-1}}\right).$$

Now if we return to our context we can use the following ramification formula from [Ben93, Thm 3.12.1]:

$$|L : K|\psi(A) - \psi(B) = \frac{1}{2} \sum_{\mathfrak{p}\in \operatorname{Spec}_1(B)} v_{\mathfrak{p}}(\mathfrak{D}_{B/A})\psi(B/\mathfrak{p}).$$

We have $\operatorname{grdim} A = \prod_{i=1}^n \frac{1}{1-t^{a_i}}$, so by computing the derivative of $(1-t)^n \operatorname{grdim} A$ at $t = 1$ we get:

$$\psi(A) = \frac{1}{\prod_i a_i} \sum_i \frac{a_i - 1}{2} \ ,$$

and similarly for $\psi(B)$. As $|L : K| = \prod a_i / \prod b_i$, we obtain:

$$|L : K|\psi(A) - \psi(B) = \frac{1}{\prod_i b_i} \sum_i \frac{a_i - b_i}{2} = \frac{1}{2\prod_i b_i} \deg J_{B/A}.$$

On the other hand, for $\mathfrak{p} \in \operatorname{Spec}_1(B)$, if d denotes the degree of a homogeneous polynomial P generating $\mathfrak{p}$, we have

$$\operatorname{grdim} B/\mathfrak{p} = (1 - t^d) \prod_i \frac{1}{1 - t^{b_i}} \ ,$$

so after computation we get

$$\psi(B/\mathfrak{p}) = \frac{d}{\prod_i b_i} \ .$$

Thus, the r.h.s. of the ramification formula becomes

$$\frac{1}{2\prod_i b_i} \sum_{P\in \operatorname{Spec}_1(B)} v_P(\theta_{B/A}) \deg P = \frac{1}{2\prod_i b_i} \deg \theta_{B/A}$$

and we can conclude that $\deg J_{B/A} = \deg \theta_{B/A}$. □

As a direct consequence of Propositions 1.5 and 1.6, we obtain the factorisation of $J_{B/A}$.

Theorem 1.8. *If $A \subseteq B$ is a finite graded polynomial extension, then we have, with the notations above:*

$$J_{B/A} \doteq \prod_{Q\in \operatorname{Spec}_1^{\mathrm{ram}}(B)} Q^{e_Q - 1}.$$

This formula settles the first part of Thm. 0.1. In section 3 we will define the well-ramified property, in order to deal with the second part of Thm. 0.1.

2. Geometric properties

In this section we recall some definitions and elementary facts about the ramification locus and branch locus of a branched covering. These will be used in section 3, in order to give geometric interpretations of the well-ramified property (Def. 3.1 and Prop. 3.2).

2.1. **Ramification locus and branch locus.** Let us define the varieties $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$, so that to the extension $A \subseteq B$ (as above), of degree r, corresponds an algebraic quasi-homogoneous finite morphism $f : V \to U$. We denote by J the Jacobian of f.

Proposition 2.1. *In V, we have equalities between:*

(i) the set of points where f is not étale, i.e. *zeros of the Jacobian J;*
(ii) the union of the sets of zeros of the ramified polynomials of B;
(iii) the set of zeros of the generator $\theta_{B/A}$ of the different $\mathfrak{D}_{B/A}$.

This set is called the *ramification locus* V_{ram}.

Proof. (ii)=(iii) comes from Prop. 1.4, and (iii)=(i) from Prop. 1.6. □

The following (well-known) proposition gives an upper bound for the cardinality of the fibers of f.

Proposition 2.2. *For all $u \in U$, $|f^{-1}(u)| \leq r$, where r is the degree of f.*

Proof. (After [Sha77, II.5.Thm.6].) Let u be in U, and write $f^{-1}(u) = \{v_1, \ldots, v_m\}$. One can easily find an element b in $B = \mathcal{O}_V$ such that all the values $b(v_i)$ are distinct.

As f is finite, B is a module of finite type (of rank r) over A. Thus, every element in B is a root of a unitary polynomial with coefficients in A, and degree less than or equal to r. Let $P \in A[T]$ be such a polynomial for b, and write $P = \sum_{i=0}^{d} a_i T^i$, with $d \leq r$ and $a_d = 1$. We have

$$\sum_{i=0}^{d} a_i b^i = 0.$$

For $j \in \{1, \ldots, m\}$, as $f(v_j) = u$, specializing the above identity at v_j gives

$$\sum_{i=0}^{d} a_i(u) b(v_j)^i = 0.$$

So the polynomial $\sum_i a_i(u) T^i$ has m distinct roots $b(v_1), \ldots, b(v_m)$, and has degree d, so we obtain $m \leq d \leq r$. □

Points in U whose fiber does not have maximal cardinality are called *branch points*, and form the *branch locus* of f in U:

$$U_{\text{branch}} := \left\{ u \in U \ , \ \left| f^{-1}(u) \right| < r \right\} \ .$$

It is easy to show that U_{branch} is closed for the Zariski topology and is not equal to U (cf. [Sha77, II.5.Thm. 7]), so that $U - U_{\text{branch}}$ is dense in U.

Proposition 2.3. *With the notations above, we have the following equality:*

$$f(V_{\text{ram}}) = U_{\text{branch}} .$$

So $V_{\text{ram}} \subseteq f^{-1}(U_{\text{branch}})$. *Moreover, the restriction of* f*:*

$$V - f^{-1}(U_{\text{branch}}) \twoheadrightarrow U - U_{\text{branch}}$$

is a topological r*-fold covering (for the transcendental topology).*

Proof. We set $U' := U - U_{\text{branch}}$ and $V' := V - f^{-1}(U_{\text{branch}})$; these are Zariski-open.

First, let u be in U_{branch}. As U' is dense in U, we can find a sequence $u^{(k)}$ of elements in U', whose limit is u. Let us write $f^{-1}(u) = \{v_1, \dots, v_p\}$ (with $p < r$), and $f^{-1}(u^{(k)}) = \{v_1^{(k)}, \dots, v_r^{(k)}\}$. Up to extracting subsequences, we can assume that each sequence $(v_i^{(k)})_{k\in\mathbb{N}}$ converges towards one of the v_j. As $r < n$, we have at least two sequences, say $v_1^{(k)}$ and $v_2^{(k)}$, whose limit is the same element of $f^{-1}(u)$, say v_1. But for all k, we have $v_1^{(k)} \neq v_2^{(k)}$ and $f(v_1^{(k)}) = f(v_2^{(k)}) = u^{(k)}$. If $J(v_1) \neq 0$, this contradicts the inverse function theorem. So $J(v_1) = 0$, and $u \in f(Z(J)) = f(V_{\text{ram}})$.

By [Sha77, II.5.3.Cor. 2], if f is unramified at u, then for all $v \in f^{-1}(u)$, the tangent mapping of f at v is an isomorphism. That means exactly that $f^{-1}(U - U_{\text{branch}}) \subseteq V - Z(J)$, *i.e.* $f(Z(J)) \subseteq U_{\text{branch}}$.

Let us fix an element u in U'. For all v in $f^{-1}(u)$, we know that $J(v) \neq 0$, so, by the inverse function theorem, there exists a neighbourhood E_v of v in V' such that the restriction $f : E_v \to f(E_v)$ is an isomorphism. Let us write $f^{-1}(u) = \{v_1, \dots, v_r\}$. Obviously we can suppose that the E_{v_i}'s are pairwise disjoint. Then $\Omega_u := f(A_{v_1}) \cap \cdots \cap f(A_{v_r})$ is a neighbourhood of u in U', and for x in $f^{-1}(\Omega_u)$ there exists a unique i such that x is in A_{v_i}. Thus we get natural map $f^{-1}(\Omega_u) \to \Omega_u \times \{1, \dots, r\}$, $x \mapsto (f(x), i)$, which is clearly a homeomorphism. □

2.2. **Ramification indices of a branched covering.** As explained in Namba's book ([Nam87, Ex. 1.1.2]), the map f is more precisely a *finite branched covering* of U (according to [Nam87, Def. 1.1.1]), of degree r.

Because of the inclusion $Z(J) = V_{\text{ram}} \subseteq f^{-1}(U_{\text{branch}})$, we know that the irreducible components of $f^{-1}(U_{\text{branch}})$ are the $Z(Q)$, for $Q \in \operatorname{Spec}_1^{\text{ram}}(B)$, plus possibly some other components associated to unramified polynomials. We recall here, for the sake of completeness, some classical properties, thanks to which the ramification indices can be interpreted via the cardinality of the fibers.

Proposition 2.4 (After Namba)**.** *Let* u *be a non-singular point of* U_{branch}*, and* $v \in f^{-1}(u)$*. Then:*

(a) v *is non-singular in* $f^{-1}(U_{\text{branch}})$*. In particular, there is a unique irreducible component* C_v *of* $f^{-1}(U_{\text{branch}})$ *containing* v*.*

(b) There exists a connected open neighbourhood Ω_v *of* v *such that, for the restriction of* f*:*

$$\widetilde{f} : \Omega_v \to f(\Omega_v) ,$$

for each u' *in* $f(\Omega_v) \cap (U - U_{\text{branch}})$*, the cardinality of the fiber* $\widetilde{f}^{-1}(u')$ *is the ramification index* $e_{\mathfrak{q}}$ *of the ideal* $\mathfrak{q} \in \operatorname{Spec}_1(B)$ *defining* C_v*.*

Proof. We refer to Theorem 1.1.8 and Corollary 1.1.13 in [Nam87]. □

3. Well-ramified extensions

3.1. **The well-ramified property.** As above we consider a finite graded polynomial extension $A \subseteq B$, given by a morphism f. We denote by J its Jacobian, and by e_Q the ramification index of a polynomial $Q \in \operatorname{Spec}_1(B)$.

Definition 3.1. We say that the extension $A \subseteq B$, or the morphism f, is *well-ramified*, if:

$$(J) \cap A = \left(\prod_{Q \in \operatorname{Spec}_1^{\mathrm{ram}}(B)} Q^{e_Q} \right) .$$

In this case we will call the polynomial $\prod Q^{e_Q}$ the *discriminant* of the extension, and denote it by $D_{B/A}$.

Note that if the extension is well-ramified, the quotient $D_{B/A}/J_{B/A}$ is exactly the product of all ramified polynomials of the extension.

Flagrantly, the definition above makes the second part of Thm. 0.1 a tautology. So our terminology might at this point seem quite mysterious. Actually the remaining of this section will be concerned with giving several equivalent characterisations of well-ramified extensions (Prop. 3.2), that ought to make this terminology (and the usefulness of this notion) much more transparent.

3.2. **Characterisations of the well-ramified property.** Most of the characterisations given below are very elementary, but are worth mentioning so as to get a full view of what is a well-ramified extension.

Proposition 3.2. *Let $A \subseteq B$ a finite graded polynomial extension, and $f : V \to U$ its associated morphism. The following properties are equivalent:*

(i) *the extension $A \subseteq B$ is well-ramified (as defined in 3.1);*

(ii) $\left(\prod_{Q \in \operatorname{Spec}_1^{\mathrm{ram}}(B)} Q \right) \cap A = \left(\prod_{Q \in \operatorname{Spec}_1^{\mathrm{ram}}(B)} Q^{e_Q} \right)$*;*

(iii) *the polynomial $\prod_{Q \in \operatorname{Spec}_1^{\mathrm{ram}}(B)} Q^{e_Q}$ lies in A;*

(iv) *for any $\mathfrak{p} \in \operatorname{Spec}_1(A)$, if there exists $\mathfrak{q}_0 \in \operatorname{Spec}_1(B)$ over $\mathfrak{p}$ which is ramified, then any other $\mathfrak{q} \in \operatorname{Spec}_1(B)$ over $\mathfrak{p}$ is also ramified;*

(v) *if P is an irreducible polynomial in A, then, as a polynomial in B, either it is reduced, or it is completely non-reduced,* i.e. *, any of its irreducible factors appears at least twice;*

(vi) $f^{-1}(U_{\mathrm{branch}}) = V_{\mathrm{ram}}$*;*

(vii) $f(V_{\mathrm{ram}}) \cap f(V - V_{\mathrm{ram}}) = \varnothing$*.*

Proof. Let $S := \prod_{Q \in \operatorname{Spec}_1^{\mathrm{ram}}(B)} Q$, and $R := \prod_{Q \in \operatorname{Spec}_1^{\mathrm{ram}}(B)} Q^{e_Q}$. From Thm. 1.8, we have also: $J = \prod_{Q \in \operatorname{Spec}_1^{\mathrm{ram}}(B)} Q^{e_Q - 1}$.

We begin with some general elementary facts. We have $(S) \cap A = \bigcap_{Q \in \operatorname{Spec}_1^{\mathrm{ram}}(B)} (Q) \cap A$. For $Q \in \operatorname{Spec}_1^{\mathrm{ram}}(B)$, denote by $\widetilde{Q}$ an irreducible in A such that Q lies over $\widetilde{Q}$ (*i.e.* $(Q) \cap A = \widetilde{Q}$). As we work in UFDs, we get that $(S) \cap A$ is principal, generated by

$$\widetilde{S} := \operatorname{lcm}\left(\widetilde{Q} \mid Q \in \operatorname{Spec}_1^{\mathrm{ram}}(B) \right) .$$

For $Q \in \mathrm{Spec}_1^{\mathrm{ram}}(B)$, Q^{e_Q} divides $\widetilde{Q}$, so R divides $\widetilde{S}$. Moreover, J divides R, so: $\widetilde{S} \subseteq (R) \cap A \subseteq (J) \cap A$. Conversely, S divides J, so: $(J) \cap A \subseteq (S) \cap A = (\widetilde{S})$.

Thus we always have $(J) \cap A = (S) \cap A$, and the statement (ii) is just an alternate definition of the well-ramified property: $(i) \Leftrightarrow (ii)$.

$(iii) \Leftrightarrow (ii)$: we have $(S) \cap A \subseteq (R)$ and $R \in (S)$. So R lies in A if and only if $(S) \cap A = (R)$.

$(v) \Leftrightarrow (iv)$, since (v) is just a "polynomial" rephrasing of (iv).

$(iv) \Leftrightarrow (iii)$: let us denote by $\mathrm{Spec}_1^{\mathrm{ram}}(A)$ the set of primes $\mathfrak{p}$ in $\mathrm{Spec}_1(A)$ such that there exists at least one prime $\mathfrak{q}$ over $\mathfrak{p}$ which is ramified. Then:

$$(*) \qquad R = \prod_{Q \in \mathrm{Spec}_1^{\mathrm{ram}}(B)} Q^{e_Q} = \prod_{P \in \mathrm{Spec}_1^{\mathrm{ram}}(A)} \prod_{\substack{Q \in \mathrm{Spec}_1^{\mathrm{ram}}(B) \\ Q \text{ over } P}} Q^{e_Q} .$$

If we suppose (iv), then whenever P is in $\mathrm{Spec}_1^{\mathrm{ram}}(A)$, all the Q over P are ramified. Thus:

$$R = \prod_{P \in \mathrm{Spec}_1^{\mathrm{ram}}(A)} \prod_{Q \text{ over } P} Q^{e_Q} = \prod_{P \in \mathrm{Spec}_1^{\mathrm{ram}}(A)} P$$

and the polynomial R lies in A.

Conversely, suppose that R lies in A. Consider P in $\mathrm{Spec}_1^{\mathrm{ram}}(A)$, and Q in $\mathrm{Spec}_1(B)$ lying over P. Then there exists Q_0 in $\mathrm{Spec}_1^{\mathrm{ram}}(B)$ such that $(Q_0) \cap A = (P) = (Q) \cap A$. As (R) is contained in $(Q_0) \cap A$, we obtain that Q also divides R, so is among the factors of the product $(*)$ above. Thus (Q) is ramified, and (iv) is verified.

$(vii) \Rightarrow (vi)$: if v lies in $f^{-1}(U_{\mathrm{branch}})$, we have $f(v) \in U_{\mathrm{branch}} = f(V_{\mathrm{ram}})$, so (vii) implies that $v \in V_{\mathrm{ram}}$. Thus $f^{-1}(U_{\mathrm{branch}}) \subseteq V_{\mathrm{ram}}$. The other inclusion is from Prop. 2.3.

$(i) \Rightarrow (vii)$: we have $(\widetilde{S}) = (J) \cap A$, so that $Z(\widetilde{S}) = f(Z(J))$. If the extension is well-ramified, we obtain: $\widetilde{S} \doteq \prod_{Q \in \mathrm{Spec}_1^{\mathrm{ram}}(B)} Q^{e_Q}$. Thus J and $\widetilde{S}$ have the same irreducible factors in B, which implies $Z(J) = f^{-1}(Z(\widetilde{S}))$. So $f(V_{\mathrm{ram}}) = Z(\widetilde{S})$, whereas $f(V - V_{\mathrm{ram}}) = U - Z(\widetilde{S})$.

$(vi) \Rightarrow (v)$: suppose that there exists Q in $\mathrm{Spec}_1^{\mathrm{ram}}(B)$, such that $\widetilde{Q}$ has one irreducible factor (in B) which is not a ramified polynomial, say M. Then we can choose v in $Z(M) - Z(J)$. As $M(v) = 0$, we have $\widetilde{Q}(f(v)) = 0$ and $\widetilde{S}(f(v)) = 0$. So $f(v) \in Z(\widetilde{S}) = f(Z(J)) = U_{\mathrm{branch}}$, which contradicts (i). ☐

3.3. **Examples and counterexamples.** A fundamental case when the extension is well-ramified is the Galois case. If $A = B^G$, with G a (reflection) group, all the ramification indices of the ideals over a prime of A are the same. In our setting, the well-ramified property is somewhat a weak version of the normality (the extension is always separable since we work in characteristic zero).

Of course the notion is strictly weaker than being a Galois extension. Take for example the extension $\mathbb{C}[X^2 + Y^3, X^2Y^3] \subseteq \mathbb{C}[X, Y]$, with Jacobian $J = 6XY^2(X^2 - Y^3)$. The ramified polynomials are (X) (index 2) and (Y) (index 3), both above (X^2Y^3), and $(X^2 - Y^3)$ (index 2), above $((X^2 - Y^3)^2)$. So this extension is well-ramified but not Galois.

A simple example of a not well-ramified extension is:

$$A = \mathbb{C}[X^2Y, X^2 + Y] \subseteq \mathbb{C}[X, Y] = B,$$

which is free of rank 4. Here the ideal (X^2Y) in A has two ideals above in B: (X) which is ramified and (Y) which is not, so the extension is not well-ramified. We compute $\theta_{B/A} = \mathrm{Jac}(f) = X(Y-X^2) = S$ (using the notations of the proof of Prop. 3.2). So $R = X^2(Y-X^2)^2$ is not in A, actually $(S) \cap A$ is generated by $X^2Y(Y-X^2)^2$.

3.4. **Applications to the Lyashko-Looijenga morphisms.** For any (well-generated) complex reflection group, Bessis introduced in [Bes07] a morphism called the Lyashko-Looijenga morphism (LL). It gives rise to a finite graded polynomial extensions as defined in Def. 1.1. In [Rip10b], using the characterisations of Prop. 3.2, we prove that the extensions LL are always well-ramified. In particular, this implies that the quotient $D_{\mathrm{LL}}/J_{\mathrm{LL}}$ of the LL-discriminant over the LL-jacobian is the product of the ramified polynomials of the extension LL. This is an important structural property, that is used in [Rip10b] to derive new combinatorial results about certain factorisations of a Coxeter element in complex reflection groups.

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